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Lyapunov Number for a Noisy 2^n Cycle

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A stochastic one-dimensional map which produces a sequence of period doubling bifurcations is theoretically studied. We obtain analytic expressions, to a second-order approximation, of the local distribution function of fluctuating orbital points and the Lyapunov number for a noisy 2^n cycle. The expressions satisfy scaling laws and well agree with the results of numerical experiments when the external noise is weak. A scaling factor for the noise level is formulated in terms of the derivatives of a deterministic map. From it, the scaling factor is refined to be 6.6190 . . . . The Lyapunov number shows that, when the external noise is weaker than some extent, the noisy orbit is more stable rather than the deterministic one.

KEY WORDS: Stochastic one-dimensional map; period doubling bifurcations; Lyapunov number; universal noise effect; universal scaling.

1. INTRODUCTION

Recently, universal scaling properties in an infinite sequence of period doubling bifurcations have been extensively studied since Feigenbaum's discovery.[1] The universalities have been confirmed by some numerical experiments on simple dynamical systems.[2-4] In real physical systems, the sequence of period doubling bifurcations have been observed[5-7]; however, the universalities have not been fully confirmed. Experimental results on the universal ratios have been somewhat different from the theoretical ones. The reason why the discrepancies appear may be that the observed ratios are not asymptotic ones. However, it is possively expected from the confirmations by the numerical experiments that the experimental evidences will be offered in a variety of fields.

In a real system, the fine structure of the bifurcation sequence is wiped out by an unavoidable external noise. Thus, we can observe at most some

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finite sequence. In order to analyze observed data, it is necessary to consider the noise effect. It becomes a significant problem how the external noise wipes out the structure. Numerical experiments on the noise effect have been carried out on stochastic one-dimensional maps.[8-10] Some of the results obtained by Crutchfield *et al.*[10] are as follows. Let us suppose that we can see up to 2^n cycle under some external noise. In order to see twice, the external noise must have a standard deviation, σ, about 6.6 times smaller. The Lyapunov number, λ, at an accumulation point, r_∞, scales with the power law such as $\lambda(r_\infty, \sigma) = A\sigma^\theta$ with $\theta = 0.37 \pm 0.01$. From the results together with the Huberman–Rudnik scaling[11] of the Lyapunov number, they suggest the existence of a homogeneous scaling function $F(\bar{r}, \sigma)$ such that $\lambda(\bar{r}, \sigma) = -(d/d\sigma)F(\bar{r}, \sigma)$ and $F(L^{a_r}\bar{r}, L^{a_\sigma}\sigma) = LF(\bar{r}, \sigma)$, where $\bar{r} \equiv (r - r_\infty)/r_\infty$. Shraiman *et al.*[12] have shown that the Lyapunov number satisfies a scaling form from a correlation-function expression. Then the results have been obtained on the assumption that there exists a fixed point of a recursion relation for a renormalized noise amplitude.

In this paper, the effect of the external noise on the period doubling phenomena to chaos is theoretically studied without ambiguous assumptions. One of our interests is how the stability of the 2^n cycle is influenced by the external noise. The other interest is how the universal features in deterministic period doubling phenomena contribute to the noise effect. As a dynamical system, we adopt a stochastic one-dimensional map which consists of a deterministic one-dimensional map and an additive random variable. It is the simplest system exhibiting the universal properties. In Section 2, we derive the local form of (2^n)th iterate of the stochastic one-dimensional map. From the local form, we estimate the noise level region in which a period is maintained as a deterministic one. In Section 3, universality of the (2^n)th iterated map is discussed. In Section 4, the distribution function of fluctuations around $x_1^{(n)}$ is obtained, where $x_1^{(n)}$ is a deterministic orbital point closest to the maximum point $\bar{x}$ of the deterministic map. The distribution function is shown to satisfy a scaling form. The universal scaling factor, β, for the noise level is formulated in terms of the derivatives of the deterministic map at the orbital points of a stable 2^n-point cycle. In Section 5, the Lyapunov number is expressed by an integral form. It becomes a simple form at a superstable point. The Lyapunov number satisfies the scaling form studied by Shraiman *et al.*[12] The effect of the external noise on the stability is discussed.

2. 2^nTH ITERATE OF A STOCHASTIC ONE-DIMENSIONAL MAP

Let us study a stochastic one-dimensional map as

$$x_{m+1} = F(x_m, r) + f_m \tag{2.1}$$

where $F(x,r)$ is a one-dimensional map exhibiting an infinite sequence of period doubling bifurcations with a control parameter r, and f_m is a Gaussian random variable with $\langle f_m \rangle = 0$ and $\langle f_m f_{m'} \rangle = \sigma^2 \delta_{mm'}$. The dynamical variable x_m ranges over the interval $[0,1]$ and the function $F(x,r)$ has a unique differentiable maximum point $\bar{x}$ of a second order. Let $x_1^{(n)}, x_2^{(n)}, \ldots, x_{2^n}^{(n)}$ be the deterministic orbital points of the 2^n-point cycle, i.e., $F(x_i^{(n)},r) = x_{i+1}^{(n)}$ for $i = 1,2,\ldots,2^n-1$ and $F(x_{2^n}^{(n)},r) = x_1^{(n)}$. The point $x_1^{(n)}$ is closest to $\bar{x}$. Let $\tilde{r}_n$ be the superstable point for the deterministic 2^n-point cycle.

We focus our attention to a 2^n cycle with the control parameter r close to $\tilde{r}_n$ under a weak external noise. The noise is supposed to be weak enough for the fluctuations to be localized around the deterministic orbital points $x_i(n)$. It is preferable to construct (2^n)th iterated map. The global form is not needed for the present weak noise case. Thus, let us derive the local form around $x_1^{(n)}$. A dynamical variable Δ_m is defined

$$\Delta_m = x_{m\times 2^n} - x_1^{(n)} \tag{2.2}$$

Let $x_{m\times 2^n}$ be the $(m \times 2^n)$th iterate of (2.1) from $x_0 = x_1^{(n)}$. The choice of the initial condition does not lose a generality for large m. Δ_{m+1} is given by

$$\begin{aligned}\Delta_{m+1} = F\Big(\ldots\Big(F\big(F(x_1^{(n)}+\Delta_m)+f_M\big)+f_{M+1}\Big)\ldots\Big)+f_{M+2^n-2}\Big)\\ +f_{M+2^n-1} - x_1^{(n)}\end{aligned} \tag{2.3}$$

where $M = m \times 2^n$. In order to get the local form, each term is expanded in powers of a deviation from each corresponding point $x_i^{(n)}$.

Since $x_1^{(n)}$ coincides with $\bar{x}$ at $r = \tilde{r}_n$, $x_1^{(n)}$ lies very much close to $\bar{x}$. Therefore, we keep the powers of the deviation Δ_m from $x_1^{(n)}$ up to the second order. In other expansions around $x_i^{(n)}$, $i \neq 1$, it may be sufficient to keep the terms up to the first order. Then, we get

$$\Delta_{m+1} = A^{(n)}\Delta_m^2 + \chi^{(n)}\Delta_m + \xi_m \tag{2.4}$$

where

$$A^{(n)} = \tfrac{1}{2}F''\big(x_1^{(n)},r\big)\prod_{i=2}^{2^n}F'\big(x_i^{(n)},r\big) \tag{2.5}$$

$$\chi^{(n)} = F^{(2^n)\prime}\big(x_1^{(n)},r\big) \tag{2.6}$$

$$\xi_m = \prod_{j=1}^{2^n}\gamma_j^{(n)} f_{m\times 2^n+j-1} \tag{2.7}$$

$$\gamma_j^{(n)} = \prod_{i=j+1}^{2^n}F'\big(x_i^{(n)},r\big) \qquad \text{for} \quad 1 \leqslant j \leqslant 2^n-1, \quad \text{and} \quad \gamma_{2^n}^{(n)} = 1 \tag{2.8}$$

where the prime denotes differentiation with respect to x. The quantity $\chi^{(n)}$ is a stability parameter which changes with r, such as $\chi^{(n)}=1$ at $r=r_n$, $\chi^{(n)}=0$ at $r=\tilde{r}_n$ and $\chi^{(n)}=-1$ at $r=r_{n+1}$, where r_n is a bifurcation point from which 2^n-point cycle appears. The variable ξ_m is a new Gaussian random variable with

$$\langle \xi_m \rangle = 0 \quad \text{and} \quad \langle \xi_m \xi_{m'} \rangle = \left(\gamma^{(n)}\sigma\right)^2 \delta_{mm'} \tag{2.9}$$

where

$$\left(\gamma^{(n)}\right)^2 = \sum_{i=1}^{2^n} \left(\gamma_i^{(n)}\right)^2 \tag{2.10}$$

Let us estimate the condition under which the period of the noisy cycle is maintained as 2^n. That is, we must study the localization condition of a time sequence $\{\Delta_0, \Delta_1, \Delta_2, \ldots\}$ with $\Delta_0 = 0$. First, we suppose $r=\tilde{r}_n$, $A^{(n)} \geqslant 0$ and $|\xi_m| \leqslant \tilde{\xi}^{(n)}$, where $\tilde{\xi}^{(n)} = (4|A^{(n)}|)^{-1}$. In $\Delta_m - \Delta_{m+1}$ plane, the iterated stochastic map (2.4) is represented by a parabola which randomly shifts up and down in the vertical direction. The parabola at every time intersects with 45° line as shown in Fig. 1. The slope of the parabola at lower intersect lies in $[1-\sqrt{2}, 1]$. Therefore, if, for some period k, the random variable ξ_m happens not to fluctuate appreciably, i.e., $\xi_{m+i} \simeq \xi'$ for $i=0,1,\ldots,k$, Δ_{m+i} tends to the lower intersect $[1-(1-4A^{(n)}\xi')^{1/2}]/(2A^{(n)})$. However, in most cases, the fluctuations of ξ_m are fully appreciable. We consider the sequence $\{\Delta_0, \tilde{\Delta}_1, \tilde{\Delta}_2, \ldots\}$ where $\tilde{\Delta}_m \equiv [1-(1-4A^{(n)}\xi_{m-1})^{1/2}]/(2A^{(n)})$. The quantity $\tilde{\Delta}_m$ is a virtual limiting point which is realized when ξ_n, $n \geqslant m$, were fixed to ξ_{m-1}. The real points Δ_m may be regarded as the first steps of virtual sequences toward $\tilde{\Delta}_m$. The distribution

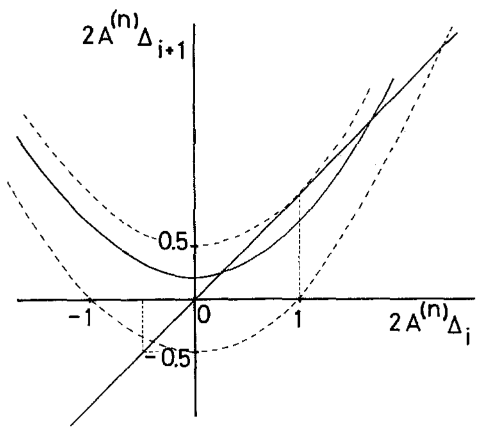


Fig. 1. Schematic drawing of (2.4) with $\chi^{(n)}=0.0$ at some instance. The dashed curves represent (2.4) with $\xi_m = \pm 1/(4A^{(n)})$.

of $\tilde{\Delta}_m$ is localized within the region $[(-\sqrt{2}+1)/(2A^{(n)}), 1/(2A^{(n)})]$. Then it can be concluded that the distribution of Δ_m is localized. Next, we suppose that ξ_m sometimes fluctuates beyond $\tilde{\xi}^{(n)}$. Whenever ξ_m fluctuates beyond $\tilde{\xi}^{(n)}$, the next step Δ_{m+1} becomes greater than Δ_m because the parabola $\Delta_{m+1} = A^{(n)}\Delta_m^2 + \xi_m$ is located above 45° line $\Delta_{m+1} = \Delta_m$. Therefore, frequent oversteppings of ξ_m cause the escape of Δ_m from the origin. Once Δ_m becomes greater than $[1+(1+4A^{(n)}\xi_{\max})^{1/2}]/(2A^{(n)})$, which is the upper intersect of the parabola shifting most downward with 45° line, every future step increasingly goes away from the origin. Thus, as long as the fluctuations of ξ_m are restricted within $[-\tilde{\xi}^{(n)}, \tilde{\xi}^{(n)}]$, the localization occurs. Here, we shall consider the Gaussian noise (2.9), so that the restriction for the maximum intensity of the noise is not satisfied. However, if the fluctuations beyond $\tilde{\xi}^{(n)}$ rarely occur, the localization must hold. Therefore, the localization condition turns out to be

$$\gamma^{(n)}\sigma \ll \frac{1}{4|A^{(n)}|} \tag{2.11}$$

We have supposed $\chi^{(n)} = 0$, but even in the case $\chi^{(n)} \neq 0$, the same can be discussed, if $|\chi^{(n)}| \ll 1$.

A global distribution consists of 2^n disjoint distributions. If the localization discussed above disappears, the disjoint distributions merge pairwise, say, a mergence phenomenon.[8] If the escape motion from $x_1^{(n)}$ occurs, Δ_m must be trapped around $F^{(2^{n-1})}(x_1^{(n)})$. In fact, when $A^{(n)}\{\gtrless\}0$, the successive points Δ_m stretch in the $\left\{\begin{smallmatrix}\text{positive}\\ \text{negative}\end{smallmatrix}\right\}$ direction and $x_{1+2^{n-1}}^{(n)}$ exists in the $\left\{\begin{smallmatrix}\text{positive}\\ \text{negative}\end{smallmatrix}\right\}$ side of $x_1^{(n)}$.

3. UNIVERSALITY OF THE (2^n)TH ITERATE OF THE STOCHASTIC MAP

It has been shown[1] that the rescaled local form of the (2^n)th iterate of $F(x,r)$ tends to a universal function as $n\to\infty$. On the basis of the universality, it is expected that (2.4) exhibits some universality as $n\to\infty$. Thus we shall discuss on the universalities of the coefficients $A^{(n)}$, $\chi^{(n)}$, and $\gamma^{(n)}$.

The stability parameter $\chi^{(n)}$ has been shown by Daido[13] to have a universal form depending upon the control parameter r. He has calculated it by a perturbation method up to a second-order approximation, i.e.,

$$\chi^{(n)} \simeq 1 - 1.8508x - 0.1492x^2 \tag{3.1}$$

where $x \equiv (r-r_n)/(r_{n+1}-r_n)$. The variable x may be replaced by

$$x \sim -\frac{r_\infty - r}{d_n} + \frac{\delta}{\delta - 1} \tag{3.2}$$

where $d_n \equiv r_{n+1} - r_n$ and $\delta \equiv \lim_{n\to\infty} d_n / d_{n+1} = 4.6692 \ldots$.[1] In the vicinity of the onset point r_∞, it is convenient to consider the dependence of $\chi^{(n)}$ on $r_\infty - r$. From (3.1) and (3.2), we get $\chi^{(n)}(r_\infty - r) = \chi^{(n+1)}((r_\infty - r)/\delta)$.

Let us consider a ratio between $A^{(n)}$ and $A^{(n+1)}$ at points, say, r and r', with corresponding stability for the 2^n- and 2^{n+1}-point cycle, i.e., $\chi^{(n)}(r) = \chi^{(n+1)}(r')$. The ratio $A^{(n+1)}/A^{(n)}$ can be written by

$$\frac{A^{(n+1)}}{A^{(n)}} = \frac{F''\left(x_1^{(n+1)}, r'\right)\chi^{(n+1)}(r')F'\left(x_1^{(n)}, r\right)}{F''\left(x_1^{(n)}, r\right)\chi^{(n)}(r)F'\left(x_1^{(n+1)}, r'\right)} \tag{3.3}$$

As n is increased, $x_1^{(n)}$ and r tend to accumulation points $\bar{x}$ and r_∞, respectively. Thus, $F''(x_1^{(n)}, r)$ becomes independent upon n as $n \to \infty$. For large enough n, the ratio (3.3) becomes $F'(x_1^{(n)}, r)/F'(x_1^{(n+1)}, r')$. Expanding the derivatives in powers of $(x_1^{(n)} - \bar{x})$ and $(x_1^{(n+1)} - \bar{x})$, we obtain

$$\frac{A^{(n+1)}}{A^{(n)}} = \frac{\left(x_1^{(n)} - \bar{x}\right) + O\left(\left(x_1^{(n)} - \bar{x}\right)^2\right)}{\left(x_1^{(n+1)} - \bar{x}\right) + O\left(\left(x_1^{(n+1)} - \bar{x}\right)^2\right)} \tag{3.4}$$

The ratio between $(x_1^{(n)} - \bar{x})$ and $(x_1^{(n+1)} - \bar{x})$ is well known to tend to a universal value $-\alpha = -2.5029 \ldots$.[1] Thus, it can be concluded that

$$\lim_{n\to\infty} \frac{A^{(n+1)}}{A^{(n)}} = -\alpha \tag{3.5}$$

Let us consider a ratio between $\gamma^{(n)}$ and $\gamma^{(n+1)}$ at points r and r' with corresponding stability for the 2^n- and 2^{n+1}-point cycle. A motion of 2^{n+1}-point cycle can be divided into two almost similar cycles of duration 2^n. It follows that

$$F'\left(x_i^{(n)}, r\right) \simeq F'\left(x_i^{(n+1)}, r'\right) \simeq F'\left(x_{i+2^n}^{(n+1)}, r'\right) \tag{3.6}$$

Since $\gamma_1^{(n)}$ can be written by $2A^{(n)}/F''(x_1^{(n)}, r)$, from (3.5) we obtain

$$\gamma_1^{(n+1)} \sim -\alpha\gamma_1^{(n)} \tag{3.7}$$

(3.6) and (3.7) lead

$$\gamma_i^{(n+1)} \simeq -\alpha\gamma_i^{(n)} \qquad \text{for} \quad 2 \leqslant i \leqslant 2^n \tag{3.8}$$

since

$$\gamma_i^{(n+1)} = \gamma_1^{(n+1)} \Big/ \prod_{j=2}^{i} F'\left(x_j^{(n+1)}, r'\right) \simeq -\alpha\gamma_1^{(n)} \Big/ \prod_{j=2}^{i} F'\left(x_j^{(n)}, r\right)$$

From (3.6), the other $\gamma_i^{(n+1)}$, $2^n + 1 \leqslant i \leqslant 2^{n+1}$, are approximately related to $\gamma_i^{(n)}$ as

$$\gamma_{i+2^n}^{(n+1)} \simeq \gamma_i^{(n)} \qquad \text{for} \quad 1 \leqslant i \leqslant 2^n \tag{3.9}$$

since

$$\gamma_{i+2^n}^{(n+1)} = \prod_{j=i+2^n+1}^{2^n} F'\left(x_j^{(n+1)}, r'\right) \simeq \prod_{j=i+1}^{2^n} F'\left(x_j^{(n)}, r\right) = \gamma_i^{(n)}$$

Then, using (3.7), (3.8), and (3.9), we find that

$$\gamma^{(n+1)} \simeq \left(1 + \alpha^2\right)^{1/2} \gamma^{(n)} \tag{3.10}$$

where $(1 + \alpha^2)^{1/2} = 2.6952\ldots$. The ratio $\gamma^{(n+1)}/\gamma^{(n)}$ has been numerically calculated for the logistic difference equation. We found that the ratio tends to the value of $2.6445\ldots$. The ratio is possibly universal for a large class of maps.

The basis for the universality is that the amplitudes $\gamma^{(n)}$ satisfy the recursion relation derived by Shraiman *et al.*[12] Regarding $\gamma^{(n)}$ as a function of $x_1^{(n)}$, n, and r, after some algebra we find the recursion relation

$$\begin{aligned} \gamma^{(n+1)}\left(x_1^{(n+1)}, r\right)^2 = &\left\{F^{(2^n)\prime}\left(F^{(2^n)}\left(x_1^{(n+1)}, r\right), r\right)\gamma^{(n)}\left(x_1^{(n+1)}, r\right)\right\}^2 \\ &+ \gamma^{(n)}\left(F^{(2^n)}\left(x_1^{(n+1)}, r\right), r\right)^2 \end{aligned} \tag{3.11}$$

The local form of $F^{(2^n)}(x, r)$ is known to tend to a universal function as $n \to \infty$.[1] Since (3.11) involves only the universal function as $n \to \infty$, the asymptotic solution must be universal. Thus, the ratio we obtained,

$$\gamma = \lim_{n \to \infty} \frac{\gamma^{(n+1)}}{\gamma^{(n)}} = 2.6445\ldots \tag{3.12}$$

must be universal.

4. DISTRIBUTION FUNCTION

Using the iterated map (2.4), let us derive the distribution function for the fluctuations of Δ_m. Since the condition (2.13) has been imposed, the points Δ_m are confined in the region $(-|\Delta^*|, |\Delta^*|)$, where $\Delta^* \equiv (2A^{(n)})^{-1}$. It is convenient to rescale Δ_m to $\Delta'_m = \Delta_m/\Delta^*$ and ξ_m to $\xi'_m = \xi_m/\Delta^*$. Then, the iterated map (2.4) is rewritten by using the new variables as

$$\Delta'_{m+1} = \tfrac{1}{2}\Delta'^2_m + \chi^{(n)}\Delta'_m + \xi'_m \tag{4.1}$$

where $\langle \xi'_m \rangle = 0$ and $\langle \xi'_m \xi'_{m'} \rangle = (2A^{(n)}\gamma^{(n)}\sigma)^2 \delta_{mm'} \equiv \epsilon^2 \delta_{mm'}$. From the condition (2.13), the standard deviation ϵ is supposed to be much less than unity. Hereafter, let us consider the standard deviation ϵ and the stability parameter $\chi^{(n)}$ as small parameters and keep only the first and second leading terms, say, a second approximation.

A time sequence $(\Delta'_0, \Delta'_1, \Delta'_2, \ldots)$ with the initial condition $\Delta'_0 = 0$ can be expressed in terms of ξ'_m under the second approximation. It follows that

$$\Delta'_0 = 0, \quad \Delta'_1 = \xi'_0, \quad \text{and} \quad \Delta'_m = \tfrac{1}{2}\xi'^2_{m-2} + \chi^{(n)}\xi'_{m-2} + \xi'_{m-1} \qquad \text{for} \quad m \geqslant 2 \tag{4.2}$$

The probability $P(\geqslant a)$ that $\Delta' \geqslant a$ is given by

$$P(\geqslant \alpha) = \iint_{(1/2)x^2 + \chi^{(n)}x + y \geqslant a} dx\, dy\; g(x) g(y) \tag{4.3}$$

where $g(x) = [(2\pi)^{1/2}\epsilon]^{-1}\exp[-x^2/(2\epsilon^2)]$. Since a probability density $\rho(a)$ is given by $-dP/da$,

$$\rho(a) = \frac{1}{2\pi\epsilon^2}\int_{-\infty}^{\infty} dx \exp\left[-\frac{\left(a - x^2/2 - \chi^{(n)}x\right)^2 + x^2}{2\epsilon^2}\right] \tag{4.4}$$

Keeping only the first and second leading terms in the exponent and performing the integration, we get

$$\rho(a) = \frac{1}{(2\pi)^{1/2}\epsilon(1-a)^{1/2}} \exp\left(-\frac{a^2}{2\epsilon^2}\right) \tag{4.5}$$

The probability density (4.5) is singular at $a = 1$; however, this is irrelevant to the present case, i.e., $\epsilon \ll 1$. Thus, expanding $1/(1-a)^{1/2}$ in powers of a, we finally obtain

$$\rho(a) = \frac{1}{(2\pi)^{1/2}\epsilon}\left(1 + \frac{a}{2}\right)\exp\left(-\frac{a^2}{2\epsilon^2}\right) \tag{4.6}$$

On the original scale, the distribution function is written as

$$\rho(\Delta) = \frac{1}{(2\pi)^{1/2}\gamma^{(n)}\sigma}\left(1 + A^{(n)}\Delta\right)\exp\left[-\frac{\Delta^2}{2\left(\gamma^{(n)}\sigma\right)^2}\right] \tag{4.7}$$

$A^{(n)}$ and $\gamma^{(n)}$ depend upon r; therefore, (4.7) may be regarded as a function of Δ, σ, and r. In the vicinity of the onset point r_∞ it is more convenient to consider the dependence on a distance $r_\infty - r$ instead of r.

The distribution function (4.7) leads to a scaling form which has been found in the numerical experiments.[10] Rescaling $r_\infty - r \to (r_\infty - r)/\delta$, the period of the cycle is doubled and the stability of the resulting cycle becomes the same as the one of the old cycle. Remembering that $A^{(n+1)}/A^{(n)}$ and $\gamma^{(n+1)}/\gamma^{(n)}$ tend to the universal values $-\alpha$ and γ, respectively, we find

$$\rho(\Delta; \sigma, r_\infty - r) = \frac{1}{\alpha}\rho\left(-\frac{\Delta}{\alpha}; \frac{\sigma}{\beta}, \frac{r_\infty - r}{\delta}\right) \tag{4.8}$$

where

$$\beta \equiv \lim_{n\to\infty} \frac{|A^{(n+1)}\gamma^{(n+1)}|}{|A^{(n)}\gamma^{(n)}|} = \alpha\gamma = 2.5029\ldots \times 2.6445\ldots$$
$$= 6.6190\ldots \tag{4.9}$$

The scaling factor β for the external noise is in good agreement with the result of numerical experiments due to Crutchfield *et al.*[10] It should be

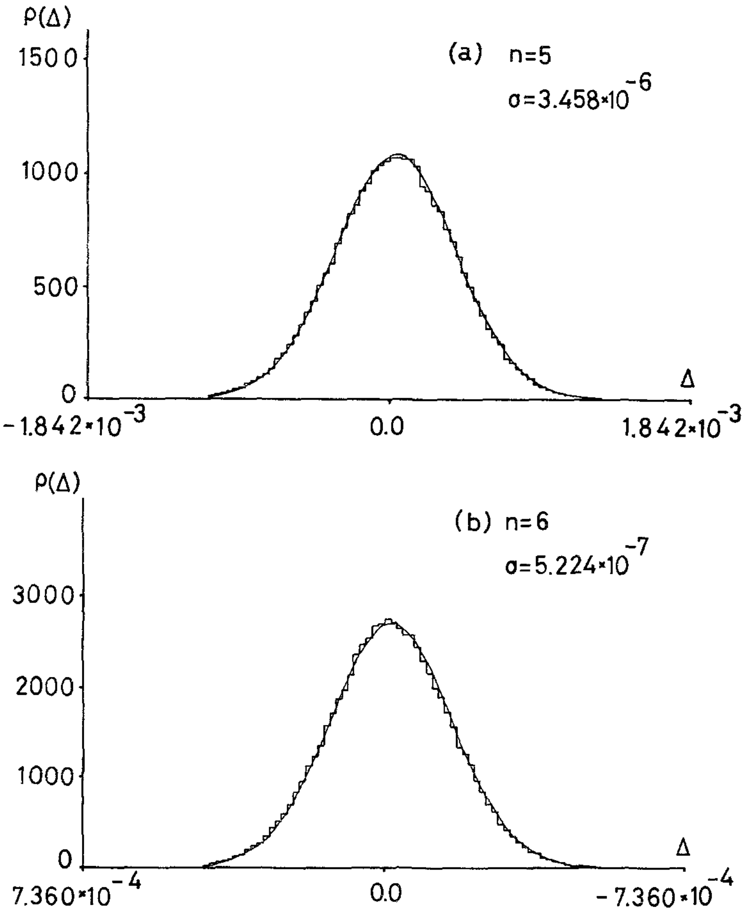


Fig. 2. Distributions of fluctuations of Δ_i. Solid curves show the theoretical distribution function (4.7). Histograms are obtained by numerical experiments. The adopted dynamical system is the logistic difference equation $x_{n+1} = rx_n(1 - x_n)$. The right end points of the abscissae are $1/(2A^{(n)})$. (a) $r = \tilde{r}_5 = 3.5692435\ldots$ (superstable point for 2^5-point cycle), $\sigma = 3.458 \times 10^{-6}$, $A^{(5)} = 271.4\ldots$, $\gamma^{(5)} = 106.5\ldots$. (b) $r = \tilde{r}_6 = 3.5697953\ldots$ (superstable point for 2^6-point cycle), $\sigma = 5.224 \times 10^{-7}$, $A^{(6)} = -679.4\ldots$, $\gamma^{(6)} = 281.8\ldots$. The abscissa is stretched by $-\alpha$ and the ordinate is reduced by α.

noted that the scaling factor is formulated in terms of the derivatives of the deterministic map.

In Fig. 2a, the distribution function (4.6) is compared to a numerically constructed histogram for the logistic difference equation with $r = \tilde{r}_5$ and $\sigma = 0.2(2A^{(5)}\gamma^{(5)})^{-1} = 3.458 \times 10^{-6}$. The figure shows good agreement. The scaling law (4.8) is also confirmed by numerical experiments. Figure 2b shows that, when the abscissa is reversed and stretched by α and the ordinate is reduced by α, the numerical constructed histogram with $r = \tilde{r}_6$ and $\sigma = 3.458 \times 10^{-6}/6.619$ is in good agreement with the one with $r = \tilde{r}_5$ and $\sigma = 3.458 \times 10^{-6}$.

5. LYAPUNOV NUMBER

The Lyapunov number, λ, which is the average of repulsion nearby orbits is a useful quantity to investigate the nature of an orbit. A positive value implies a repeller, i.e., initial separations exponentially grow, a vanishing value implies a marginal situation, i.e., initial separations neither grow nor decay, and a negative value implies an attractor. In order to investigate the noise effect on the stability, let us derive an analytic expression of the Lyapunov number.

The Lyapunov number is expressed by

$$\lambda(r, \sigma) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \ln|F'(x_i, r)| \tag{5.1}$$

where $\{x_i\}$ is produced by (2.1). The control parameter r has been supposed to be close to $\tilde{r}_n$. We partition the time sequence $\{x_i\}$ with $x_0 = x_1^{(n)}$ by 2^n cycles as $\{x_0, x_1, \ldots, x_{2^n-1}\}, \{x_{2^n}, x_{2^n+1}, \ldots, x_{2^{n+1}-1}\}, \ldots$. The summation in (5.1) is first carried out over 2^n points of each cycle as

$$\lambda(r, \sigma) = \lim_{M \to \infty} \frac{1}{M} \sum_{i=1}^{M} \frac{1}{2^n} \ln|\chi_i^{(n)}| \tag{5.2}$$

where $M \equiv N/2^n$ and

$$\chi_i^{(n)} \equiv \sum_{j=0}^{2^n-1} F'(x_{(i-1)2^n+j}, r)$$

Remembering that $x_{(i-1)2^n} = x_1^{(n)} + \Delta_i$ and that $x_1^{(n)}$ is very much close to $\bar{x}$, a leading term approximation leads to

$$\chi_i^{(n)} = 2A^{(n)}\Delta_i + \chi^{(n)} \tag{5.3}$$

where $A^{(n)}$ and $\chi^{(n)}$ are defined by (2.5) and (2.6). A higher series $B^{(n)}\Delta_i^2 + C^{(n)}\Delta_i^3$, where $B^{(n)} \equiv \frac{1}{2}F'''(x_1^{(n)}, r)\gamma_1^{(n)}$, $C^{(n)} \equiv \frac{1}{6}F''''(x_1^{(n)}, r) \times \gamma_1^{(n)}, \ldots,$ is of the order of ϵ^2/α^n, because $\gamma_1^{(n)} \propto \alpha^n$ and Δ_i is of the order of $\gamma^{(n)}\sigma = \epsilon/(2A^{(n)}) \propto \epsilon\alpha^{-n}$. Therefore, the higher series does not contribute as $n \to \infty$. The point $x_{1+2^{n-1}}^{(n)}$ is the second nearest point from $\bar{x}$, so the second leading term for $\chi_i^{(n)}$ can be written by

$$\left(F'(x_1^{(n)}, r) + F''(x_1^{(n)}, r)\Delta_i\right) \prod_{i=2}^{2^n} F'(x_i^{(n)}, r) \cdot F''(x_{1+2^{n-1}}^{(n)}, r)\delta x_{i,1+2^{n-1}}$$

$$= \left(\chi^{(n)} + 2A^{(n)}\Delta_i\right) \frac{F''\left(x_{1+2^{n-1}}^{(n)}, r\right)}{F'\left(x_{1+2^{n-1}}^{(n)}, r\right)} \delta x_{i,1+2^{n-1}} \tag{5.4}$$

where $\delta x_{i,1+2^{n-1}} \equiv x_{(i-1)2^n+2^{n-1}+2} - x_{1+2^{n-1}}^{(n)}$. $\delta x_{i,1+2^{n-1}}$ is approximately of the order of $\gamma^{(n-1)}\sigma$ and $F'(x_{1+2^{n-1}}^{(n)}, r)$ is approximately of the order of $\alpha/\gamma_1^{(n-1)} \simeq F''(\bar{x}, r)\alpha/(2A^{(n-1)})$, because $\gamma_1^{(n)} \simeq \gamma_1^{(n-1)}F'(x_{1+2^{n-1}}^{(n)}, r)\gamma_1^{(n-1)}$, which is followed by the definition. Therefore, (5.4) turns out to be of the order of $(\chi^{(n)} + \epsilon)2A^{(n-1)}\gamma^{(n-1)}\sigma/\alpha \simeq (\chi^{(n)} + \epsilon)\epsilon/\alpha\beta$, where $\alpha\beta \simeq 17$. It is concluded from the estimation that the formula (5.3) is sufficient for the calculation of the Lyapunov number up the second-order approximation.

(5.2) is also expressed by the alternative form as

$$\lambda(r, \sigma) = \lim_{M \to \infty} \frac{1}{M} \sum_{i=1}^{M} \frac{1}{2^n} \ln\left|\frac{d\Delta_{i+1}}{d\Delta_i}\right| = \lim_{M \to \infty} \frac{1}{2^n M} \ln\left|\frac{d\Delta_M}{d\Delta_1}\right| \tag{5.5}$$

because $d\Delta_{i+1}/d\Delta_i = 2A^{(n)}\Delta_i + \chi^{(n)}$, which is obtained by differentiating the iterated map $\Delta_{i+1} = A^{(n)}\Delta_i^2 + \chi^{(n)}\Delta_i + \xi_i$ under a fixed ξ_i. From (5.5), it seems that $|d\Delta_M| = \exp[2^n M\lambda(r, \sigma)]|d\Delta_1|$, which clarifies the physical meaning of the Lyapunov number.

Using the distribution function (4.7) for Δ_i, the summation in (5.2) can be transformed into the integration form as

$$\lambda(r, \sigma) = \frac{1}{2^n} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}\gamma^{(n)}\sigma} \ln|2A^{(n)}\Delta + \chi^{(n)}|$$

$$\times (1 + A^{(n)}\Delta)\exp\left[-\frac{\Delta^2}{2(\gamma^{(n)}\sigma)^2}\right] d\Delta \tag{5.6}$$

Putting $y \equiv \Delta/(\gamma^{(n)}\sigma)$, we get the integration form of the Lyapunov number as

$$\lambda(r, \sigma) = \frac{1}{2^n} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} \ln|\epsilon y + \chi^{(n)}|\left(1 + \frac{\epsilon y}{2}\right)\exp\left(-\frac{y^2}{2}\right) dy \tag{5.7}$$

where $\epsilon = 2A^{(n)}\gamma^{(n)}\sigma$.

At a superstable point $r = \tilde{r}_n$ the formula is simply expressed in terms of σ as

$$\lambda(\tilde{r}_n, \sigma) = \frac{1}{2^n} (\ln|2A^{(n)}\gamma^{(n)}\sigma| + C) \tag{5.8}$$

where $C \equiv (2\pi^{1/2})^{-1}\int_{-\infty}^{\infty}\ln|y|\exp(-y^2/2)\,dy = -0.635\ldots$. When $2|A^{(n)}|\gamma^{(n)}\sigma \lesssim 0.5$, the analytic form (5.8) is in good agreement with the numerical results as shown in Fig. 3a.

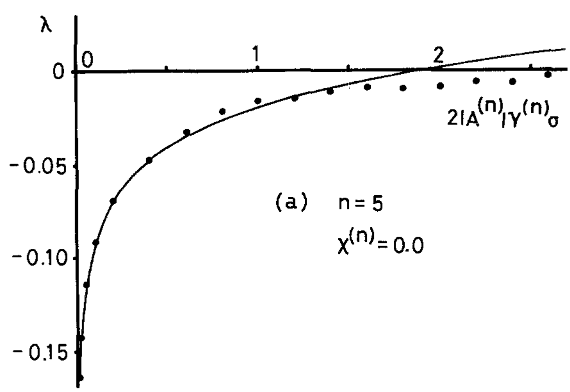
When $r \neq \tilde{r}_n$ and $|\epsilon| \ll |\chi^{(n)}|$, (5.7) can be written by

$$\lambda(r,\sigma) = \frac{1}{2^n}\int_{-\infty}^{\infty}\frac{1}{(2\pi)^{1/2}}\left[\ln|\chi^{(n)}| + \frac{\epsilon y}{\chi^{(n)}} - \frac{1}{2}\left(\frac{\epsilon y}{\chi^{(n)}}\right)^2\right]$$

$$\times\left(1+\frac{\epsilon y}{2}\right)\exp\left(-\frac{y^2}{2}\right)dy$$

$$= \lambda(r,0) - \frac{1}{2^{n+1}}(1-\chi^{(n)})\left(\frac{\epsilon}{\chi^{(n)}}\right)^2. \tag{5.9}$$

It follows that the Lyapunov number decreases when a sufficiently weak noise is added. That is, it shows that, if the external noise level σ is weaker than about $|\chi^{(n)}/(2A^{(n)}\gamma^{(n)})|$, the noisy orbit is more stable than the deterministic one. This seems to be strange. However, the following consideration[10] clarifies the effect. The noisy orbit may be regarded as a wandering motion over points on the attractors at adjacent control parameter values. The effect of fluctuations is to average the structure of deterministic attractors over some range of nearby parameters. The Lyapunov number can be obtained by the average of λ in the deterministic limit as $\lambda(r,0) + d\lambda/dr\langle\delta r\rangle + \frac{1}{2}d^2\lambda/dr^2\langle\delta r^2\rangle$, where $\langle\cdots\rangle$ denotes averaging and δr is the parameter deviation from r. Under a sufficiently weak noise, $\langle\delta r\rangle$ vanishes and $\langle\delta r^2\rangle$ is proportional to σ^2.[10] The coefficent $\frac{1}{2}d^2\lambda/dr^2$ is negative. Then we find that $\lambda(r,\sigma) - \lambda(r,0) \propto \frac{1}{2}d^2\lambda/dr^2\sigma^2 < 0$.[14] The convexity of the curve $\lambda(r,0)$ results from the existence of the superstability,



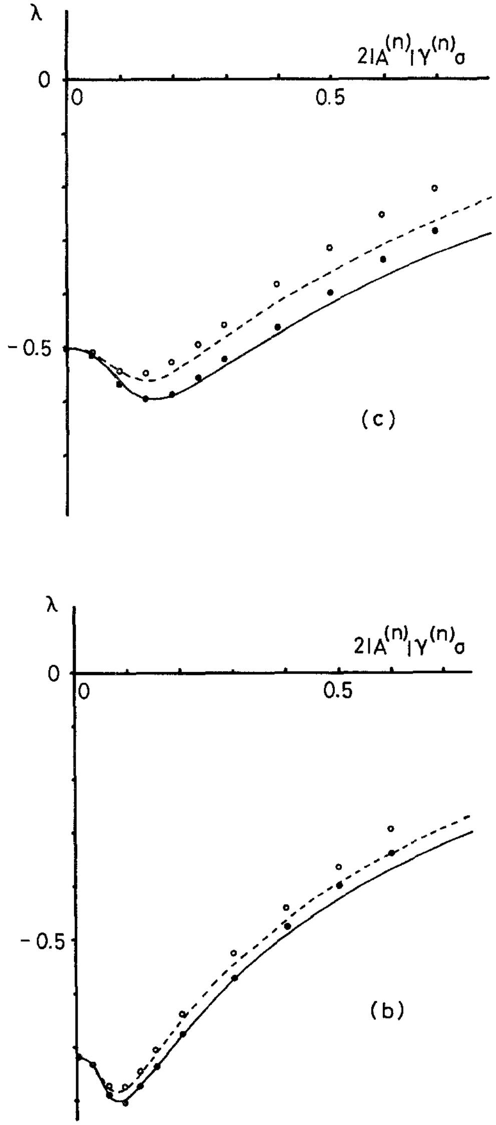


Fig. 3. Lyapunov number vs external noise level. The curves represent the theoretical function (5.7). The circles are the results of numerical experiments. The adopted dynamical system is the logistic difference equation $x_{n+1} = rx_n(1 - x_n)$ and the control parameter r is changed in the 2^5-cycle region. (a) $r = \tilde{r}_5 = 3.5692435\ldots$ (superstable point), $\chi^{(5)} = 0.0$, $A^{(5)} = 271.4\ldots$, $\gamma^{(5)} = 106.5\ldots$. (b) ----, O; $r = 3.5691968\ldots$, $\chi^{(5)} = 0.1$, $A^{(5)} = 263.9\ldots$, $\gamma^{(5)} = 104.2\ldots$. ——, ●; $r = 3.5692898\ldots$, $\chi^{(5)} = -0.1$, $A^{(5)} = 278.6\ldots$, $\gamma^{(5)} = 108.8\ldots$. (c) ----, O; $r = 3.5691497\ldots$, $\chi^{(5)} = 0.2$, $A^{(5)} = 256.1\ldots$, $\gamma^{(5)} = 101.8\ldots$; ——, ●; $r = 3.5693358\ldots$, $\chi^{(5)} = -0.2$, $A^{(5)} = 285.4\ldots$, $\gamma^{(5)} = 110.9\ldots$.

i.e., $\lambda(\tilde{r}_n, 0) = -\infty$. Therefore, this noise effect also occurs in dynamical systems exhibiting the superstability within finite parameter regions.

The behaviors of (5.7) with $\chi^{(n)} = \pm 0.1$ and ± 0.2 are shown in Figs. 3b and 3c. They are in good agreement with the results of numerical experiments in the region of $2|A^{(n)}|\gamma^{(n)}\sigma \lesssim 0.5$. Increasing the noise level stronger than $(4|A^{(n)}|\gamma^{(n)})^{-1}$, the discrepancy between the formula (5.7) and the results of numerical experiments becomes noticeable. The mergence phenomenon has been observed at $2|A^{(n)}|\gamma^{(n)}\sigma \simeq 0.5 \sim 0.6$ in our numerical experiments. This shows that our premise fails when $2|A^{(n)}|\gamma^{(n)}\sigma \geq 0.5$. The numerical results are consistent with the localization condition (2.11).

The Lyapunov number in the vicinity of the onset point r_∞ satisfies the universal scaling theory which has been found by Shraiman *et al.*[12] Rescaling $r_\infty - r \to (r_\infty - r)/\delta$ and $\sigma \to \sigma/\beta$, from (5.7) and (4.9) we also obtain the scaling form

$$\lambda(r_\infty - r, \sigma) = 2\lambda\left(\frac{r_\infty - r}{\delta}, \frac{\sigma}{\beta}\right) \tag{5.10}$$

6. CONCLUSIONS

We have studied the effect of weak external noise. Our theory does not involve the mergence phenomenon, which is one of the features observed in noisy period doubling phenomena. The limitation concerning the noise level, however, makes the discussions clear. Without ambiguous assumptions, we have expressed the Lyapunov number in terms of the noise level and the control parameter. The expression satisfies the scaling form.

We have formulated the scaling factor β for the noise level in terms of the derivatives of the deterministic map. From it we can refine the value and show the value to be universal. The universality is caused by the universal structure of (2^n)th iterated deterministic map. Therefore, the universal effect of noise appears for the weak noise which does not destroy the universal structure of the deterministic map.

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